

Review Exercises for Chapter 4

Compute the sums in Exercises 1–8.

1. $\sum_{i=1}^4 i^2$.
2. $\sum_{j=1}^3 j^3$.
3. $\sum_{i=1}^5 \frac{2^i}{i(i+1)}$.
4. $\sum_{j=4}^8 \frac{j^2 - 10}{3j}$.
5. $\sum_{i=1}^{500} (3i + 7)$.
6. $\sum_{i=n}^{n+3} \frac{i^2 - 1}{i + 1}$ (n is a non-negative integer).
7. $\sum_{i=0}^{10} [(i+1)^4 - i^4]$.
8. $\sum_{i=2}^{60} \left(\frac{1}{i} - \frac{1}{i-1} \right)$.
9. Let f be defined on $[0, 1]$ by

$$f(x) = \begin{cases} 1, & 0 \leq x < \frac{1}{3}, \\ 2, & \frac{1}{3} \leq x < \frac{1}{4}, \\ 3, & \frac{1}{4} \leq x < \frac{1}{3}, \\ 4, & \frac{1}{3} \leq x < \frac{1}{2}, \\ 5, & \frac{1}{2} \leq x \leq 1. \end{cases}$$

Find $\int_0^1 f(x) dx$.

10. Let f be defined by

$$f(x) = \begin{cases} -1, & -1 \leq x < 0, \\ 2, & 0 \leq x < 1, \\ 3, & 1 \leq x \leq 2. \end{cases}$$

Find $\int_{-1}^2 f(x) dx$.

11. Interpret the integral in Exercise 9 in terms of distances and velocities.
 12. Interpret the integral in Exercise 10 in terms of distances and velocities.
- Evaluate the definite integrals in Exercises 13–16.

13. $\int_3^5 (-2x^3 + x^2) dx$.
14. $\int_1^3 \frac{x^3 - 5}{x^2} dx$.
15. $\int_1^2 \frac{\frac{1}{3}s^2 - (s^4 + 1)}{2s^2} ds$.
16. $\int_1^2 \frac{x^2 + 3x + 2}{x + 1} dx$.

Find the area under the graphs of the functions between the indicated limits in Exercises 17–20.

17. $y = x^3 + x^2, 0 \leq x \leq 1$.
18. $y = \frac{x^2 + 2x + 1}{x^4}, 1 \leq x \leq 2$.
19. $y = (x + 3)^{4/3}, 0 \leq x \leq 2$.
20. $y = (x - 1)^{1/2}, 1 \leq x \leq 2$.

21. (a) Find upper and lower sums for

$$\int_0^1 \frac{4}{1 + x^2} dx$$

within 0.2 of one another. (b) Look at the average of these sums. Can you guess what the exact integral is?

22. Find upper and lower sums for $\int_2^3 \frac{1}{x} dx$ within $\frac{1}{10}$ of one another.

23. (a) Find $\frac{d}{dx} \left[\frac{1}{(1 + x^2)} \right]$.

(b) Find the area under the graph of the function $x/(1 + x^2)^2$, from $x = 0$ to $x = 1$.

24. Find $(d/dx)[x^3/(1 + x^3)]$. (b) Find the area under the graph of $x^2/(1 + x^3)^2$ from $x = 1$ to $x = 2$.

25. Find the area under the graph of $y = mx + b$ from $x = a_1$ to $x = a_2$ and verify your answer by using plane geometry. Assume that $mx + b \geq 0$ on $[a_1, a_2]$.

26. Find the area under the graph of the function $y = (1/x^2) + x + 1$ from $x = 1$ to $x = 2$.

27. Find the area under the graph of $y = x^2 + 1$ from -1 to 2 and sketch the region.

28. Find the area under the graph of

$$f(x) = \begin{cases} -x^3 & \text{if } x \leq 0, \\ x^3 & \text{if } x > 0 \end{cases}$$

from $x = -1$ to $x = 1$ and sketch.

29. (a) Verify the integration formula

$$\int \frac{x^2}{(x^3 + 6)^2} dx = \frac{1}{12} \left[\frac{x^3 + 2}{(x^3 + 6)} \right] + C.$$

(b) Find the area under the graph of $y = x^2/(x^3 + 6)^2$ between $x = 0$ and $x = 2$.

30. Find the area between the graphs of $y = x^3$ and $y = 5x^2 + 2x$ between $x = 0$ and $x = 2$. Sketch.

31. The curves $y = x^6 - 3$ and $y = -x^2 - 1$ divide the plane into five regions, one of which is bounded. Find its area.

32. Find the area of each of the numbered regions in Fig. 4.R.1.

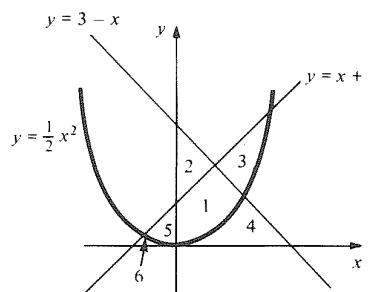


Figure 4.R.1. Find the area of the numbered regions.

33. Find the area of the region bounded by the graphs $x = y^2 - 6$ and $x = y$.
34. Find the area of the region bounded by the graphs $y = x^2 - 2$ and $y = 2 - x^2$.
35. An object is thrown at $t = 0$ from an airplane, and it has vertical velocity $v = -10 - 32t$ feet per second at time t . If the object is still falling after 10 seconds, what can you say about the altitude of the plane at $t = 0$?
36. Suppose the velocity of an object at position x is x^n , where n is some integer $\neq 1$. Find the time required to travel from $x = \frac{1}{1000}$ to $x = 1$.
37. (a) At time $t = 0$, a container has 1 liter of water in it. Water is poured in at the rate of $3t^2 - 2t + 3$ liters per minute ($t =$ time in minutes). If the container has a leak which can drain 2 liters per minute, how much water is in the container at the end of 3 minutes?
 (b) What if the leak is 4 liters per minute?
 (c) What if the leak is 8 liters per minute?
 [Hint: What happens if the tank is empty for a while?]
38. Water is poured into a container at a rate of t liters per minute. At the same time, water is leaking out at the rate of t^2 liters per minute. Assume that the container is empty at $t = 0$.
 (a) When does the amount of water in the container reach its maximum?
 (b) When is the container empty again?
39. Suppose that a supply curve $p = S(x)$ and a demand curve $p = D(x)$ are graphed and that there is a unique point (a, b) at which supply equals demand ($p =$ price/unit in dollars, $x =$ number of units). The (signed) area enclosed by $x = 0$, $x = a$, $p = b$, and $p = D(x)$ is called the *consumer's surplus* or the *consumer's loss* depending on whether the sign is positive or negative. Similarly, the (signed) area enclosed by $x = 0$, $x = a$, $p = b$, and $p = S(x)$ is called the *producer's surplus* or the *producer's loss* depending on whether the sign is positive or negative.
 (a) Let $D(x) \geq b$. Explain why the *consumer's surplus* is $\int_0^a [D(x) - b] dx$.
 (b) Let $S(x) \leq b$. Explain why the *producer's surplus* is $\int_0^a [b - S(x)] dx$.
 (c) "If the price stabilizes at \$6 per unit, then some people are still willing to pay a higher price, but benefit by paying the lower price of \$6 per unit. The total of these benefits over $[0, a]$ is the consumer's surplus." Explain this in the language of integration.
 (d) Find the consumer's and producer's surplus for the supply curve $p = x^2/8$ and the demand curve $p = -(x/4) + 1$.
40. The demand for wood products in 1975 was about 12.6 billion cubic feet. By measuring order

increases, it was determined that x years after 1975, the demand increased by $9x/1000$; that is, $D'(x) = 9x/1000$, where $D(x)$ is the demand x years after 1975, in billions of cubic feet.

- (a) Use the fundamental theorem of calculus to show that

$$D(x) = D(0) + \int_0^x (9t/1000) dt.$$

- (b) Find $D(x)$.
 (c) Find the demand for wood in 1982.
41. Suppose that an object on the x axis has velocity $v = t^2 - 4t - 5$. How far does it travel between $t = 0$ and $t = 6$?
42. Show that the actual distance travelled by a bus with velocity $v = f(t)$ is $\int_a^b |f(t)| dt$. What condition on $v = f(t)$ means that the bus made a round trip between $t = a$ and $t = b$?
43. A rock is dropped off a bridge over a gorge. The sound of the splash is heard 5.6 seconds after the rock was dropped. (Assume the rock falls with velocity $32t$ feet per second and sound travels at 1080 feet per second.)
 (a) Show by integration that the rock falls $16t^2$ feet after t seconds, and that the sound of the splash travels $1080t$ feet in t seconds.
 (b) The time T required for the rock to hit the water must satisfy $16T^2 = 1080(5.6 - T)$, because the rock and the sound wave travel equal distances. Find T .
 (c) Find the height of the bridge.
 (d) Find the number of seconds required for the sound of the splash to travel from the water to the bridge.
44. The current $I(t)$ and charge $Q(t)$ at time t (in amperes and coulombs, respectively) in a circuit are related by the equation $I(t) = Q'(t)$.
 (a) Given $Q(0) = 1$, use the fundamental theorem of calculus to justify the formula $Q(t) = 1 + \int_0^t I(r) dr$.
 (b) The voltage drop V (in volts) across a resistor of resistance R ohms is related to the current I (in amperes) by the formula $V = RI$. Suppose that in a simple circuit with a resistor made of nichrome wire, $V = 4.36$, $R = 1$, and $Q(0) = 1$. Find $Q(t)$.
 (c) Repeat (b) for a circuit with a 12-volt battery and 4-ohm resistance.
45. A ruptured sewer line causes lake contamination near a ski resort. The concentration $C(t)$ of bacteria (number per cubic centimeter) after t days is given by $C'(t) = 10^3(t - 7)$, $0 \leq t \leq 6$, after treatment of the lake at $t = 0$.
 (a) An inspector will be sent out after the bacteria concentration has dropped to half its original value $C(0)$. On which day should the inspector be sent if $C(0) = 40,000$?
 (b) What is the total change in the concentration from the fourth day to the sixth day?

46. A baseball is thrown vertically upwards from the ground with an initial upward velocity of 50 feet per second. How far has it travelled when it strikes the ground?

47. Let

$$g(y) = \begin{cases} y, & 0 \leq y < 1, \\ 2, & 1 \leq y < 2, \\ y, & 2 \leq y \leq 4. \end{cases}$$

Compute $\int_0^1 g(y) dy + \int_3^4 g(y) dy$.

48. Let

$$y(t) = \begin{cases} t, & 2 \leq t < 3, \\ -4, & 3 \leq t < 4, \\ 1, & 4 \leq t \leq 5. \end{cases}$$

Compute $\int_5^2 y(t) dt$.

49. Calculate $\frac{d}{dx} \int_0^x \frac{s^2}{1+s^3} ds$.

50. Calculate $\frac{d}{dx} \int_2^x \frac{t^3}{1-t^4} dt$.

51. Find the area between the graph of the function in Exercise 47 and the x axis.

★52. Find $\frac{d}{ds} \int_0^{s^2} \frac{1}{\sqrt[3]{x^2+1}} dx$. (*Hint*: See Exercise 43, Section 4.5.)

★53. Find $\frac{d}{dt} \int_0^{t^2+2} \frac{1}{y^2+1} dy$.

★54. Find $\int_a^b (3x^2 + x) dx$ “by hand.”

★55. Let f be defined on $[0, 1]$ by

$$f(x) = \begin{cases} 0, & x = 0, \\ 1/\sqrt{x}, & 0 < x \leq 1. \end{cases}$$

(a) Show that there are no upper sums for f on $[0, 1]$, and hence that f is not integrable.

(b) Show that every number less than 2 is a lower sum. [*Hint*: Use step functions which are zero on an interval $[0, \epsilon)$ and approximate f very closely on $[\epsilon, 1]$. Take ϵ small and use the integrability of f on $[\epsilon, 1]$.]

(c) Show that no number greater than or equal to 2 is a lower sum. [*Hint*: Show $\epsilon f(\epsilon) + \int_\epsilon^1 f(x) dx < 2$ for all ϵ in $(0, 1)$.]

(d) If you had to assign a value to $\int_0^1 f(x) dx$, what value would you assign?

★56. Modeling your discussion after the preceding exercise, find the upper or lower sums for each of the following functions on $[0, 1]$:

(a) $f(x) = \begin{cases} 0, & x = 0, \\ -1/\sqrt[3]{x}, & x > 0; \end{cases}$

(b) $f(x) = \begin{cases} 0, & x = 0, \\ 1/x^2, & x > 0. \end{cases}$